

Numerical invariants of Fano 4-folds

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Let X be a (smooth, complex) Fano 4-fold. As usual, we denote by $\mathcal{N}_1(X)$ the vector space of one-cycles in X , with real coefficients, modulo numerical equivalence; its dimension is the Picard number ρ_X of X , which coincides with the second Betti number.

For any prime divisor $D \subset X$, let $\mathcal{N}_1(D, X)$ be the linear subspace of $\mathcal{N}_1(X)$ generated by classes of curves contained in D ; its dimension is at most ρ_D . We consider the following invariant of X :

$$c_X := \max \{ \text{codim } \mathcal{N}_1(D, X) \mid D \text{ a prime divisor in } X \}.$$

Notice that $c_X \in \{0, \dots, \rho_X - 1\}$, and $c_X \geq \rho_X - \rho_D$ for any prime divisor $D \subset X$. This invariant has been introduced in [3] for Fano manifolds of arbitrary dimension; it turns out that c_X is always at most 8 [3, Th. 3.3]. Here we consider the case where X has dimension 4.

If $X = S_1 \times S_2$ is a product of Del Pezzo surfaces with $\rho_{S_1} \geq \rho_{S_2}$, then $\rho_{S_1} = c_X + 1$, $\rho_{S_2} \leq c_X + 1$, and $\rho_X = \rho_{S_1} + \rho_{S_2} \leq 2c_X + 2$ (see [3, Ex. 3.1]). In particular, as Del Pezzo surfaces have Picard number at most 9, we get $c_X \leq 8$ and $\rho_X \leq 18$.

When X is not a product of surfaces, we have the following.

Theorem 1 ([3], Th. 1.1 and Cor. 1.3). *Let X be a Fano 4-fold which is not a product of surfaces. Then $c_X \leq 3$.*

Moreover if $c_X = 3$, then $\rho_X \in \{5, 6\}$ and X has a flat fibration onto \mathbb{P}^2 , $\mathbb{P}^1 \times \mathbb{P}^1$, or the Hirzebruch surface \mathbb{F}_1 .

In this paper we consider the case $c_X = 2$, in which we give the following bound on the Picard number.

Theorem 2. *Let X be a Fano 4-fold with $c_X = 2$. Then $\rho_X \leq 12$.*

Moreover if $\rho_X \geq 7$, then there is a diagram:

$$X \longrightarrow X_1 \xrightarrow{h} \tilde{X}_1 \longrightarrow Y$$

where all the varieties are smooth and projective, $X \rightarrow X_1$ is the blow-up of a smooth irreducible surface contained in $\text{dom}(h)$, X_1 is Fano, h is birational and an isomorphism in codimension 1, and $\tilde{X}_1 \rightarrow Y$ is an elementary contraction and a conic bundle.

The author does not know whether the bound $\rho_X \leq 12$ given above is sharp. Indeed, if X is a product of surfaces with $c_X = 2$, then $\rho_X \leq 2c_X + 2 = 6$. On the other hand, all known examples of Fano 4-folds which are not products of surfaces have Picard number at most 6; it would be interesting to have examples with larger Picard number.

The main motivation for this work is the following conjecture, which remains open in the case $c_X \leq 1$.

Conjecture 3. *Let X be a Fano 4-fold. Then $\rho_X \leq 18$, with equality only if X is a product of surfaces.*

Let us recall that after boundedness, the Picard number of Fano manifolds has a maximal value in each dimension. As noticed by J. Kollár [7, Rem. 1.6], an explicit bound on ρ_X can be obtained with the same techniques used to prove boundedness. Indeed there are explicit bounds for the Betti numbers of a smooth projective variety, embedded in a projective space, in terms of the dimension and of the degree, as shown recently by F. L. Zak [11]. However such a bound on ρ_X is (conjecturally) far from being sharp: for instance, in dimension 4 we get $\rho_X < 3^{85} 5^{304} 23^4 26$. We refer the reader to Rem. 12 for more details.

The proof of Theorem 2 relies on both [3, 4]. In particular, we use the following result.

Proposition 4 ([4], Prop. 5.1). *Let X be a Fano 4-fold with $\rho_X \geq 6$ and $c_X = 2$. Then one of the following holds:*

- (i) $\rho_X \leq 12$, and there is a diagram $X \rightarrow X_1 \xrightarrow{h} \tilde{X}_1 \rightarrow Y$ as in Th. 2;
- (ii) there exists a Fano 4-fold X_2 and $\sigma: X \rightarrow X_2$ a blow-up of two disjoint smooth irreducible surfaces.

The proof of Prop. 4 is based on a construction which depends on a prime divisor $D \subset X$ with $\text{codim } \mathcal{N}_1(D, X) = 2$. This construction yields two possible outputs, giving cases (i) and (ii) above. Our strategy to prove Th. 2 is to exploit the freedom in the choice of D : we show that if every prime divisor D with $\text{codim } \mathcal{N}_1(D, X) = 2$ leads to case (ii), and $\rho_X \geq 7$, then we get a contradiction.

We use the same techniques introduced in [3], where the case $c_X \geq 3$ is studied in arbitrary dimension. The case $c_X = 2$ is more difficult, and this is one reason for which we need to work in dimension 4.

We conclude this introduction by noting that the bound $\rho_X \leq 12$ in Th. 2 follows from the geometric description of the case $\rho_X \geq 7$. Indeed the existence of the rational map $X_1 \dashrightarrow Y$ yields $\rho_{X_1} \leq 11$ by [4, Th. 1.1], and hence $\rho_X \leq 12$. The lack of analogous results in higher dimensions is one of the obstructions to study the case $c_X = 2$ in general.

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Notation and terminology

For any projective variety X , we denote by $\mathcal{N}_1(X)$ (respectively, $\mathcal{N}^1(X)$) the vector space of one-cycles (Cartier divisors), with real coefficients, modulo numerical equivalence. We denote by $[C]$ (respectively, $[D]$) the numerical equivalence class of a curve C (of a Cartier divisor D). Moreover, $\text{NE}(X) \subset \mathcal{N}_1(X)$ is the convex cone generated by classes of effective curves.

For any closed subset $Z \subset X$, we denote by $\mathcal{N}_1(Z, X)$ the subspace of $\mathcal{N}_1(X)$ generated by classes of curves contained in Z .

If D is a Cartier divisor in X , we set $D^\perp := \{\gamma \in \mathcal{N}_1(X) \mid D \cdot \gamma = 0\}$.

If X is a normal projective variety, a *contraction* of X is a surjective morphism $\varphi: X \rightarrow Y$, with connected fibers, where Y is normal and projective. The contraction is elementary if $\rho_X - \rho_Y = 1$.

Let now X be a Fano 4-fold. Then contractions of X are in bijection with faces of $\text{NE}(X)$, and elementary contractions of X correspond to one-dimensional faces of $\text{NE}(X)$, also called *extremal rays*.

Let R be an extremal ray of $\text{NE}(X)$, and $\varphi: X \rightarrow Y$ the associated contraction. If D is a divisor in X , the sign of $D \cdot R$ is the sign of $D \cdot C$, C a curve with class in R . We set $\text{Locus}(R) := \text{Exc}(\varphi)$, the locus where φ is not an isomorphism. We say that R is of type (a, b) where $a = \dim \text{Exc}(\varphi)$ and $b = \dim \varphi(\text{Exc}(\varphi))$.

We say that R type $(3, 2)^{sm}$ if Y is smooth and φ is the blow-up of a smooth, irreducible surface in Y . We recall the following very useful property.

Remark 5 ([10], Th. 1.2). Let X be a Fano 4-fold, R be an extremal ray of $\text{NE}(X)$, and $\varphi: X \rightarrow Y$ the associated contraction. If φ is birational and has fibers of dimension ≤ 1 , then R is of type $(3, 2)^{sm}$. In particular, any small extremal ray of $\text{NE}(X)$ is of type $(2, 0)$.

Preliminary results

In this section we gather some technical results needed for the proof of Th. 2.

Remark 6. Let X be a Fano manifold, $Z \subset X$ a closed subset, $\varphi: X \rightarrow Y$ a contraction, and $\varphi_*: \mathcal{N}_1(X) \rightarrow \mathcal{N}_1(Y)$ the push-forward of one-cycles. Then $\mathcal{N}_1(\varphi(Z), Y) = \varphi_*(\mathcal{N}_1(Z, X))$, so we have:

$$(7) \quad \dim \mathcal{N}_1(\varphi(Z), Y) \geq \dim \mathcal{N}_1(Z, X) - \dim \ker \varphi_*.$$

Lemma 8. *Let X be a Fano 4-fold with $\rho_X \geq 6$ and $c_X \leq 2$.*

- (a) *Let $E \subset X$ be a prime divisor which is a smooth \mathbb{P}^1 -bundle with fiber $f \subset E$, such that $E \cdot f = -1$. Then $R := \mathbb{R}_{\geq 0}[f]$ is an extremal ray of type $(3, 2)$, it is the unique extremal ray having negative intersection with E , and the target of the contraction of R is Fano.*
- (b) *Suppose that X has two extremal rays R_1 and R_2 of type $(3, 2)$, with loci E_1 and E_2 respectively, such that $E_1 \neq E_2$. Then either $E_1 \cdot R_2 > 0$ and $E_2 \cdot R_1 > 0$, or $E_1 \cdot R_2 = E_2 \cdot R_1 = 0$.*

Proof. Statement (a) follows from [4, Rem. 5.3].

To show (b), assume by contradiction that $E_1 \cdot R_2 > 0$ and $E_2 \cdot R_1 = 0$. The two divisors E_1 and E_2 intersect along a surface, because $E_1 \cdot R_2 > 0$. Since $E_2 \cdot R_1 = 0$, the contraction of R_1 sends $E_1 \cap E_2$ to a curve, and by (7) this yields $\dim \mathcal{N}_1(E_1 \cap E_2, X) \leq 2$.

On the other hand, since $E_1 \cdot R_2 > 0$, $E_1 \cap E_2$ meets every non-trivial fiber of the contraction $\varphi: X \rightarrow Y$ of R_2 . This means that $\varphi_*(\mathcal{N}_1(E_2, X)) = \varphi_*(\mathcal{N}_1(E_1 \cap E_2, X))$, and hence $\mathcal{N}_1(E_2, X) = \mathcal{N}_1(E_1 \cap E_2, X) + \mathbb{R}R_2$. Thus $\dim \mathcal{N}_1(E_2, X) \leq 1 + \dim \mathcal{N}_1(E_1 \cap E_2, X) \leq 3$ and $c_X \geq \rho_X - 3 \geq 3$, a contradiction. ■

Lemma 9. *Let X be a Fano 4-fold with $\rho_X \geq 6$ and $c_X = 2$. Let $D \subset X$ be a prime divisor with $\text{codim } \mathcal{N}_1(D, X) = 2$. Then there exists an extremal ray R_1 of X , of type $(3, 2)^{sm}$, such that $D \cdot R_1 > 0$ and $R_1 \notin \mathcal{N}_1(D, X)$.*

Proof. By [3, Prop. 2.5], there exists a prime divisor $E_1 \subset X$ which is a smooth \mathbb{P}^1 -bundle with fiber $f_1 \subset E_1$, such that $E_1 \cdot f_1 = -1$, $D \cdot f_1 > 0$, and $[f_1] \notin \mathcal{N}_1(D, X)$. By Lemma 8(a), $R_1 := \mathbb{R}_{\geq 0}[f_1]$ is an extremal ray of type $(3, 2)$. Since $D \cdot R_1 > 0$, every non-trivial fiber of the contraction of R_1 must intersect D . On the other hand $R_1 \notin \mathcal{N}_1(D, X)$, thus every such fiber must have dimension 1, and R_1 is of type $(3, 2)^{sm}$ (see Rem. 5). ■

Lemma 10. *Let X be a Fano 4-fold with $\rho_X \geq 6$ and $c_X = 2$, and assume that X does not satisfy the statement of Theorem 2.*

Then for any prime divisor $D \subset X$ with $\text{codim } \mathcal{N}_1(D, X) = 2$, and for any extremal ray R_1 as in Lemma 9, there exists a second extremal ray R_2 of type $(3, 2)^{sm}$, with the following properties, where $E_i := \text{Locus}(R_i)$ for $i = 1, 2$:

- (a) $E_1 \cap E_2 = \emptyset$;
- (b) $R_1 + R_2$ is a face of $\text{NE}(X)$, whose contraction $\sigma: X \rightarrow X_2$ is the smooth blow-up of two disjoint irreducible surfaces in X_2 , and X_2 is Fano;
- (c) $D \cdot R_i > 0$ and $R_i \notin \mathcal{N}_1(D, X)$ (in particular $D \neq E_i$), for $i = 1, 2$;
- (d) $\text{codim } \mathcal{N}_1(E_i, X) = 2$, $\mathcal{N}_1(D \cap E_i, X) = \mathcal{N}_1(D, X) \cap \mathcal{N}_1(E_i, X)$, and $\text{codim } \mathcal{N}_1(D \cap E_i, X) = 3$, for $i = 1, 2$.

Proof. We use the construction of a special Mori program for $-D$ introduced in [3, §2]; we refer the reader to *loc. cit.* and references therein for the terminology. A special Mori program is just a Mori program where all involved extremal rays have positive intersection with the anticanonical divisor, see [3, Prop. 2.4].

Let $\sigma_0: X \rightarrow X_1$ be the contraction of R_1 ; notice that X_1 is Fano by Lemma 8(a). By [3, Prop. 2.4], we can consider a special Mori program for the divisor $-\sigma_0(D)$ in X_1 . Together with σ_0 , this yields a special Mori program for $-D$ in X , where the first step is precisely σ_0 .

We now apply [4, proof of Prop. 5.1] to this special Mori program. Since we are excluding by assumption case (i) of Prop. 4, we know that there exist a smooth Fano 4-fold X_2 and a contraction $\sigma: X \rightarrow X_2$ which is the blow-up of two disjoint smooth irreducible surfaces. More precisely, the proof of Prop. 4 shows that σ is the contraction of $R_1 + R_2$, where R_2 is an extremal ray of type $(3, 2)^{sm}$, with locus E_2 , such that $E_1 \cap E_2 = \emptyset$ and $D \cdot R_2 > 0$. In particular we have (a) and (b).

Fix $i \in \{1, 2\}$. By [3, Lemma 3.1.8] we have $\text{codim } \mathcal{N}_1(E_i, X) = 2$ and $\text{codim } \mathcal{N}_1(D \cap E_i, X) = 3$. Notice that $\mathcal{N}_1(D, X) \neq \mathcal{N}_1(E_i, X)$, for instance because $\mathcal{N}_1(E_i, X)$ is contained in $(E_{3-i})^\perp$, while $\mathcal{N}_1(D, X)$ is not. On the other hand $\mathcal{N}_1(D \cap E_i, X) \subseteq \mathcal{N}_1(D, X) \cap \mathcal{N}_1(E_i, X)$, and looking at dimensions we see that equality holds. So we have (d).

Finally $R_2 \notin \mathcal{N}_1(D \cap E_2, X)$ by [3, Rem. 3.1.3(2)], hence $R_2 \notin \mathcal{N}_1(D, X)$ by (d), and we have (c). ■

Lemma 11. *Let X be a Fano 4-fold with $\rho_X \geq 6$ and $c_X = 2$, and assume that X does not satisfy the statement of Theorem 2.*

Let R and R' be two extremal rays of X of type $(3, 2)^{sm}$, and set $E := \text{Locus}(R)$ and $E' := \text{Locus}(R')$. Suppose that $\text{codim } \mathcal{N}_1(E, X) = \text{codim } \mathcal{N}_1(E', X) = 2$, $E \cdot R' > 0$, $E' \cdot R > 0$, $R \notin \mathcal{N}_1(E', X)$, and $R' \notin \mathcal{N}_1(E, X)$.

Let S be an extremal ray different from R and R' . If the contraction of S is not finite on $E \cup E'$, then $E \cdot S = E' \cdot S = 0$.

Proof. We notice first of all that, since $S \neq R$ and $S \neq R'$, we have $E \cdot S \geq 0$ and $E' \cdot S \geq 0$ by Lemma 8(a).

Let us assume that the contraction of S is not finite on E , and let $C \subset E$ be an irreducible curve with $[C] \in S$.

By Lemma 10, applied with $D = E$ and $R_1 = R'$, there exists an extremal ray R'' , of type $(3, 2)^{sm}$ and with locus E'' , such that:

$$E' \cap E'' = \emptyset, \quad E \cdot R'' > 0, \quad \text{and} \quad R'' \notin \mathcal{N}_1(E, X).$$

We have $E'' \cdot R > 0$ by Lemma 8(b), moreover $S \neq R''$ because $S = \mathbb{R}_{\geq 0}[C] \subset \mathcal{N}_1(E, X)$. Therefore $E'' \cdot S \geq 0$ by Lemma 8(a).

Let $f \subset E$ be an irreducible curve with class in R . Since $E'' \cdot R > 0$, we know after [9] (see [3, Rem. 3.1.3(3)]) that there exists an irreducible curve C_1 , contained in $E \cap E''$, such that $C \equiv \lambda f + \mu C_1$, where $\lambda, \mu \in \mathbb{Q}$, and $\mu \geq 0$.

Thus $\lambda[f] + \mu[C_1] = [C] \in S$, and since S is an extremal ray of $\text{NE}(X)$ and $[f] \notin S$ (for $R \neq S$), we must have $\lambda \leq 0$. On the other hand since $C_1 \subset E''$ and $E' \cap E'' = \emptyset$, we have $E' \cdot C_1 = 0$, and hence

$$E' \cdot C = \lambda E' \cdot f \leq 0$$

(recall that $E' \cdot f > 0$ by assumption). This implies that $E' \cdot S = 0$, $\lambda = 0$, and $[C_1] \in S$.

Therefore we have shown that if the contraction of S is not finite on E , then $E' \cdot S = 0$.

Moreover, since $E' \cdot R > 0$, $E'' \cdot R > 0$, and $E'' \cdot S \geq 0$, we can repeat the same argument with the roles of R' and R'' interchanged, to find $C \equiv \lambda_2 f + \mu_2 C_2$, where $\lambda_2, \mu_2 \in \mathbb{Q}$, $\mu_2 \geq 0$, and C_2 is an irreducible curve contained in $E \cap E'$. In the same way we conclude that $[C_2] \in S$. This means that the contraction of S is not finite on E' neither, thus $E \cdot S = 0$ by what precedes. \blacksquare

Proof of Theorem 2

Let us assume that $\rho_X \geq 7$. Then X cannot be a product of surfaces, for otherwise $\rho_X \leq 2c_X + 2 = 6$ (see on p. 1).

We proceed by contradiction, and suppose that X does not satisfy the statement.

Step 1. *There exist three extremal rays R_0, R_1, R_2 , of type $(3, 2)^{sm}$, such that if $E_i := \text{Locus}(R_i)$ for $i = 0, 1, 2$, we have the following properties:*

- (a) $\text{codim } \mathcal{N}_1(E_i, X) = 2$ for $i = 0, 1, 2$;
- (b) the divisors E_0, E_1, E_2 are distinct, and $E_1 \cap E_2 = \emptyset$;
- (c) $R_1 + R_2$ is a face of $\text{NE}(X)$, whose contraction $\sigma: X \rightarrow X_2$ is the smooth blow-up of two disjoint irreducible surfaces in X_2 , and X_2 is Fano;
- (d) $E_0 \cdot R_i > 0$ and $E_i \cdot R_0 > 0$ for $i = 1, 2$;
- (e) $R_0 \notin \mathcal{N}_1(E_i, X)$ and $R_i \notin \mathcal{N}_1(E_0, X)$ for $i = 1, 2$;
- (f) $\text{codim } \mathcal{N}_1(E_0 \cap E_i) = 3$ for $i = 1, 2$.

Proof of step 1. Since $c_X = 2$, there exists some prime divisor $D \subset X$ with $\text{codim } \mathcal{N}_1(D, X) = 2$. Lemma 9 yields the existence of an extremal ray R_0 , of type $(3, 2)^{sm}$, such that $D \cdot R_0 > 0$ and $R_0 \notin \mathcal{N}_1(D, X)$. Then Lemma 10(d) implies, in particular, that $\text{codim } \mathcal{N}_1(E_0, X) = 2$, where $E_0 := \text{Locus}(R_0)$.

Now we apply Lemmas 9 and 10 to the divisor E_0 . We deduce the existence for two extremal rays R_1 and R_2 , of type $(3, 2)^{sm}$, with loci E_1 and E_2 respectively, such that (a), (b), (c), and (f) hold, and moreover $E_0 \cdot R_i > 0$, $R_i \notin \mathcal{N}_1(E_0, X)$, and $\mathcal{N}_1(E_0 \cap E_i, X) = \mathcal{N}_1(E_0, X) \cap \mathcal{N}_1(E_i, X)$, for $i = 1, 2$.

Fix $i \in \{1, 2\}$. We have $E_i \cdot R_0 > 0$ by Lemma 8(b), hence (d) holds. Moreover [3, Rem. 3.1.3(2)] yields $R_0 \notin \mathcal{N}_1(E_0 \cap E_i, X) = \mathcal{N}_1(E_0, X) \cap \mathcal{N}_1(E_i, X)$, therefore $R_0 \notin \mathcal{N}_1(E_i, X)$, and (e) holds too. \blacksquare

Consider the blow-up $\sigma: X \rightarrow X_2$ given by step 1(c). Let R' be an extremal ray of $\text{NE}(X_2)$ such that $\sigma(E_0) \cdot R' > 0$, and let $\varphi: X_2 \rightarrow Y$ be the associated contraction. We have $\rho_Y = \rho_X - 3 \geq 4$, so that $\dim Y \geq 2$.

$$X \xrightarrow{\sigma} X_2 \xrightarrow{\varphi} Y$$

Step 2. *The case where φ is birational.*

We have $R' = \sigma_*(R_3)$, R_3 an extremal ray of $\text{NE}(X)$ (see [2, §2.5]); in particular $R_3 \neq R_1$ and $R_3 \neq R_2$. Since $\sigma(\text{Locus}(R_3)) \subseteq \text{Locus}(R') \subsetneq X_2$, the contraction of R_3 is birational. Notice also that $R_3 \neq R_0$, for otherwise the locus of R' should be $\sigma(E_0)$, which is excluded because $\sigma(E_0) \cdot R' > 0$. By Lemma 8(a) we know that $E_i \cdot R_3 \geq 0$ for $i = 0, 1, 2$.

We have

$$\sigma^*(\sigma(E_0)) = E_0 + m_1 E_1 + m_2 E_2$$

with $m_i > 0$ (since $E_0 \cdot R_i > 0$ by step 1(d)) for $i = 1, 2$, and $\sigma^*(\sigma(E_0)) \cdot R_3 > 0$ by the projection formula, therefore at least one of the intersections $E_0 \cdot R_3$, $E_1 \cdot R_3$, $E_2 \cdot R_3$ is positive.

Up to exchanging R_1 and R_2 , we can assume that the intersections $E_0 \cdot R_3$ and $E_1 \cdot R_3$ are not both zero. Applying Lemma 11 with $R = R_0$, $R' = R_1$, and $S = R_3$ (notice that the assumptions are satisfied by step 1(a),(d),(e)), we see that the contraction of R_3 must be finite on $E_0 \cup E_1$. On the other hand every curve with class in R_3 must intersect $E_0 \cup E_1$, because $(E_0 + E_1) \cdot R_3 > 0$. We conclude that the contraction of R_3 has fibers of dimension at most 1, therefore R_3 is of type $(3, 2)^{sm}$ (see Rem. 5). Set $E_3 := \text{Locus}(R_3)$.

As the extremal rays R_0, R_1, R_2, R_3 are distinct, by Lemma 8(a) we know that also the divisors E_0, E_1, E_2, E_3 are distinct, and that $E_3 \cdot R_1 \geq 0$ and $E_3 \cdot R_2 \geq 0$.

Suppose that $E_1 \cdot R_3 > 0$ and $E_3 \cdot R_1 > 0$, and let $f_3 \subset X$ be a curve with class in R_3 . Using the projection formula one easily sees that $\sigma(E_3) \cdot \sigma_*(f_3) \geq 0$. On the other hand $[\sigma_*(f_3)] \in R'$, and R' is divisorial with locus $\sigma(E_3)$, so we have a contradiction.

We conclude by Lemma 8(b) that $E_1 \cdot R_3 = E_3 \cdot R_1 = 0$. Moreover, since we are assuming that the intersections $E_0 \cdot R_3$ and $E_1 \cdot R_3$ are not both zero, we must have $E_0 \cdot R_3 > 0$.

If $E_1 \cap E_3 \neq \emptyset$, then E_1 should contain some curve with class in R_3 , which is impossible because the contraction of R_3 is finite on E_1 . Thus $E_1 \cap E_3 = \emptyset$.

Similarly, applying Lemma 11 with $R = R_0$, $R' = R_2$, and $S = R_3$ (again, the assumptions are satisfied by step 1), we conclude that the contraction of R_3 is finite on E_2 too, that $E_2 \cdot R_3 = E_3 \cdot R_2 = 0$, and finally that $E_2 \cap E_3 = \emptyset$.

Therefore E_1, E_2, E_3 are pairwise disjoint \mathbb{P}^1 -bundles, with fibers $f_i \subset E_i$, such that $E_i \cdot f_i = -1$ and $E_0 \cdot f_i > 0$, for $i = 1, 2, 3$. Now [3, Lemma 3.1.7] yields that $\mathcal{N}_1(E_0 \cap E_1, X) \subseteq E_i^\perp$ for $i = 1, 2, 3$.

Applying Lemma 10 to the divisor E_1 and the extremal ray R_0 , we find a prime divisor E_4 such that $E_4 \neq E_1$, $E_0 \cap E_4 = \emptyset$, and $E_1 \cap E_4 \neq \emptyset$. Since E_2 and E_3 are disjoint from E_1 , we also have $E_4 \neq E_2$ and $E_4 \neq E_3$, so that $E_4 \cdot f_i \geq 0$ for $i = 1, 2, 3$.

For every curve $C \subset E_0 \cap E_1$ we have $C \cap E_4 = \emptyset$, therefore $E_4 \cdot C = 0$; this gives $\mathcal{N}_1(E_0 \cap E_1, X) \subseteq E_4^\perp$.

Summing-up, we have $\mathcal{N}_1(E_0 \cap E_1, X) \subseteq E_1^\perp \cap E_2^\perp \cap E_3^\perp \cap E_4^\perp$. Since $\text{codim } \mathcal{N}_1(E_0 \cap E_1, X) = 3$ by step 1(f), we deduce that $[E_1], [E_2], [E_3], [E_4]$ are linearly dependent in $\mathcal{N}^1(X)$, thus there exist rational numbers a_i , not all zero, such that $\sum_{i=1}^4 a_i E_i \equiv 0$. Fix $j \in \{1, 2, 3\}$. Intersecting with f_j we get $a_j = a_4 E_4 \cdot f_j$. In particular $a_4 \neq 0$, and we get:

$$(E_4 \cdot f_1)E_1 + (E_4 \cdot f_2)E_2 + (E_4 \cdot f_3)E_3 + E_4 \equiv 0.$$

Since a non-zero effective divisor cannot be numerically trivial, we have a contradiction.

Step 3. *The case where φ is of fiber type.*

Suppose first that Y is a surface.

Recall that $\sigma(E_1)$ and $\sigma(E_2)$ are the two surfaces blown-up by σ . Fix $i \in \{1, 2\}$. By (7) we have $\dim \mathcal{N}_1(\varphi(\sigma(E_i)), Y) \geq \dim \mathcal{N}_1(E_i, X) - 3 = \rho_X - 5 \geq 2$. Therefore $\varphi(\sigma(E_i))$ cannot be a point nor a curve, and we conclude that $\sigma(E_i)$ dominates Y under φ .

Consider now a general fiber $F \subset X_2$ of φ . Then $\dim \mathcal{N}_1(F, X_2) = 1$, and F intersects both $\sigma(E_1)$ and $\sigma(E_2)$. The inverse image $\sigma^{-1}(F)$ is a general fiber of the composition $\varphi \circ \sigma: X \rightarrow Y$, and it contains curves with class in R_1 and in R_2 , so that $\dim \mathcal{N}_1(\sigma^{-1}(F), X) = 3 = \dim \ker(\varphi \circ \sigma)_*$. This means that $\varphi \circ \sigma$ is a “quasi-elementary” contraction (see [2, Def. 3.1]), and by [2, Th. 1.1] X is a product of surfaces, a contradiction.

Suppose now that $\dim Y = 3$, so that Y is factorial with isolated canonical singularities (this is well-known – see for instance [2, Lemma 3.10(i)] and references therein). Since $\sigma(E_0) \cdot R' > 0$, the divisor $\sigma(E_0)$ intersects every fiber of φ , and we have $(\varphi \circ \sigma)(E_0) = Y$. Hence $\varphi \circ \sigma$ is generically finite on E_0 , and cannot contract the fibers f_0 of the \mathbb{P}^1 -bundle structure on E_0 given by the contraction of R_0 . The images $(\varphi \circ \sigma)(f_0)$ give a covering and unsplit family of rational curves in Y (see [1] for the terminology), and by [1, Cor. 1] there exists an elementary contraction $\psi: Y \rightarrow Z$ which contracts the curves $(\varphi \circ \sigma)(f_0)$. In particular ψ is of fiber type, and $\rho_Z = \rho_Y - 1 \geq 3$, hence Z is a surface.

$$X \xrightarrow{\sigma} X_2 \xrightarrow{\varphi} Y \xrightarrow{\psi} Z$$

Let $F \subset X_2$ be a general fiber of $\psi \circ \varphi: X_2 \rightarrow Z$. Then $\dim \mathcal{N}_1(F, X_2) = 2$, and F contains some curve of the type $\sigma(f_0) \subset \sigma(E_0)$. Notice that the surfaces $\sigma(E_1)$ and $\sigma(E_2)$ both intersect $\sigma(f_0)$, because $E_1 \cdot R_0 > 0$ and $E_2 \cdot R_0 > 0$ by step 1(d).

The inverse image $\sigma^{-1}(F)$ is a general fiber of the composition $\xi := \psi \circ \varphi \circ \sigma: X \rightarrow Z$. Since $\sigma(E_1)$ and $\sigma(E_2)$ both intersect F , $\sigma^{-1}(F)$ contains curves with class in R_1 and in R_2 , so that $\mathcal{N}_1(\sigma^{-1}(F), X) \supseteq \ker \sigma_*$. As $\sigma_*(\mathcal{N}_1(\sigma^{-1}(F), X)) = \mathcal{N}_1(F, X_2)$, we get $\dim \mathcal{N}_1(\sigma^{-1}(F), X) = 2 + \dim \mathcal{N}_1(F, X_2) = 4$. On the other hand $\dim \ker \xi_* = 4$, and as in the previous case, this means that ξ is a “quasi-elementary” contraction. Therefore by [2, Th. 1.1] X is a product of surfaces, which is again a contradiction. This concludes the proof of Th. 2. ■

Remark 12. Let X be a Fano manifold of dimension n , and assume that X is embedded in a projective space with degree d . Then it follows from [11] that

$$(13) \quad \rho_X < (n^2 + n + 2) d.$$

Indeed let us consider the three classes μ_0 , μ_1 , and μ_2 of X . These are projective invariants of an embedded variety (see [11] for the definition), in particular $\mu_0 = d$ and $\mu_1 = 2d + 2g - 2$, where g is the sectional genus of X . Using the fact that X is Fano, one can easily check that $2g - 2 < (n - 1)d$ and hence $\mu_1 < (n + 1)d$. Now it follows from [11, Cor. 1.13 and Th. 2.9(ii)] that

$$\rho_X \leq \mu_2 + \mu_0 \leq \frac{\mu_1^2}{d} + d < (n^2 + 2n + 2) d.$$

On the other hand there are explicit constants δ_n and m_n , depending only on n , such that $-m_n K_X$ is very ample, and $(-K_X)^n \leq \delta_n$ (see [5, §5.9] and references therein). Therefore X can always be embedded with degree at most $m_n^n \delta_n$, and (13) yields $\rho_X < (n^2 + 2n + 2)m_n^n \delta_n$.

In dimension 4 we can take $\delta_4 = 3^8 5^{304}$ [5, Th. 5.18]. Moreover $-4K_X$ is free [6], hence $-23K_X$ is very ample, see [8, Ex. 1.8.23]. This yields

$$\rho_X < 3^8 5^{304} 23^4 26$$

for any Fano 4-fold X . Notice that even if we take $m_4 = 5$ (as predicted by Fujita's conjecture) and $\delta_4 = 800$ (which, to the author's knowledge, is the maximal anticanonical degree among the known examples of Fano 4-folds), we still get a rather large bound.

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